Characterization of segment and convexity preserving maps

Zsolt Páles *

Abstract

Let X and Y be (real) linear spaces. For $a, b \in X$ (or $a, b \in Y$) the closed segment $[a, b]$ and the open segment $]a, b[$ connecting the points a and b are defined by

$$[a, b] := \{ta+(1-t)b \mid 0 \leq t \leq 1\}, \qquad]a, b[:= \{ta+(1-t)b \mid 0 < t < 1\}.$$

Given a convex subset $D \subset X$ and a map $f : D \to Y$, we can consider two *convexity preserving properties* for f. We say that f preserves convexity if $f(K)$ is convex for all convex subset $K \subseteq D$. Analogously, we say that f^{-1} preserves convexity or f is inversely convexity preserving if $f^{-1}(K)$ is convex whenever K is a convex subset of $f(D)$. It is immediate to see that f is convexity preserving if and only if

$$[f(x), f(y)] \subseteq f([x, y]) \qquad (x, y \in D).$$

On the other hand, f is inversely convexity preserving if and only if

$$[f(x), f(y)] \supseteq f([x, y]) \qquad (x, y \in D).$$

Functions enjoying both of the above properties, i.e., satisfying

$$f([x, y]) = [f(x), f(y)] \qquad (x, y \in D),$$

are called *segment preserving maps*. Therefore, f is segment preserving if and only if it is convexity and also inversely convexity preserving.

If the closed segments are replaced by open segments in the above definitions then we speak about *strict* convexity and segment preserving properties for f. Clearly, strict convexity or segment preserving maps are always convexity or segment preserving (in the corresponding sense), the converse, however, may not be valid.

The obvious examples for (strict) convexity and segment preserving maps are *affine maps*, i.e., functions of the form $f(x) = A(x)+a$, where $A : X \to$

*Institute of Mathematics, University of Debrecen, pales@science.unideb.hu

Y is linear and a is a constant vector. It is a natural question if there exist other types of segment preserving functions.

One can trivially see that if X and Y are equal to the set of real numbers, then $f : X \to Y$ is segment preserving if and only if it is continuous and either increasing or decreasing; furthermore, f is strictly segment preserving if and only if it is continuous and either strictly increasing, or strictly decreasing, or constant. Therefore, for the first sight, the class of such maps seems to be even more complicated in the higher-dimensional setting. However, as our main result shows, if the range of f is at least two dimensional, then the description is easier: strict segment preserving maps, moreover strict inversely convexity preserving maps can always be expressed as the ratio of an Y-valued and a real-valued affine map.